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Massless picture, massive picture, and symmetry in the Gaussian renormalization group

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Abstract: We study transformation properties of momentum space renormalization group transformations. The renormalization group transformations are composed of an integral over a high momentum field with a scale transformation of a low momentum field. As an example, we consider perturbations of a massless real euclidean free field ϕ with a smooth momentum space regulator. We show that this renormalization group admits two equivalent formulations called the massless picture and the massive picture. In the massive picture, the renormalization group is shown to have a symmetry. The symmetry consists of global (space-time independent) scale transformations of the field composed with a certain (gaussian) integral. We then translate the symmetry back to the massless picture. The relation between the symmetry and the notion of an anomalous dimension is briefly discussed.

Keywords: Renormalization Regularization and Renormalons, Field Theories in Lower Dimensions, Stochastic Processes.

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Contents

1. Introduction

Momentum space renormalization group transformations [1] have a wide range applications, both in statistical physics and in euclidean quantum field theory. We will restrict our attention to a basic class of models, to perturbations of a massless one component real scalar field ϕ on euclidean space-time $\mathbb{R}^D$. The momentum space renormalization group comes in two equivalent versions, a differential type and a discrete type. The dynamical equation is a functional differential equation in the former and a functional integral in the latter case. We will display our main formulas parallel in both versions. As a very successfull application, we mention the elegant proofs of perturbative renormalizability [2, 3]. A key idea of the renormalization group is to sort models into universality classes. The universality classes are understood in terms of fixed points and their invariant manifolds, which are properties of the transformation rather than properties of a specific bare action. But there are as many renormalization groups as there are ways to regularize a field theory. Therefore, a natural problem is to search for relations among different renormalization groups. Although this question is central in the understanding of universality, it has received little attention in the literature. See however [4]. The subject of this paper is a particular relation, the symmetry associated with global scale transformations. Although this paper will remain on a formal level, I will borrow convenient notation from mathematical renormalization theory, as for instance [5]. Since it may be unusual in field theory, I will briefly explain my notation together with the setup.

Transformations of the following kind are tailor made to study perturbations of free massless fields. We consider effective field theories, with a unit ultraviolet cutoff,[1] defined by an effective action $S(\phi) = T(\phi) + V(\phi)$, which consists of a kinetic term $T(\phi)$ and an interaction $V(\phi)$ The kinetic term is a quadratic form

$$T(\phi) \;=\; \frac{1}{2}\,\left(\phi, \frac{1}{C_\infty}\,\phi\right) \tag{1.1}$$

and is given by a free propagator C_∞. We build the ultraviolet cutoff into C_∞.[2] We will keep C_∞ fixed. The renormalization group will transform only $V(\phi)$. Any action can be written in this form, at the expense that the interaction $V(\phi)$ will in general contain another quadratic form

$$V(\phi) \;=\; \frac{1}{2}\int \mathrm{d}^D x_1 \int \mathrm{d}^D x_2 \; V_2(x_1,x_2)\;\phi(x_1)\,\phi(x_2) + \text{higher vertices}\,, \tag{1.2}$$

which corrects C_∞. Of course, it will in general also contain higher vertices, for instance a ϕ^4-interaction. To avoid confusion on an important point, let me emphasize that the interaction is understood as an effective action and will in general include non-local

[1] We use dimensionless units. The field ϕ is always scaled such as to have a unit ultraviolet cutoff

[2] The subscript $_\infty$ indicates that C_∞ is a massless propagator without infrared cutoff.

vertices. Effective actions are generated from bare actions by iterated renormalization group transformations.

The Schwinger functions of this effective field theory are generated by the functional integral

$$\exp\bigl(W(J)\bigr) = \frac{\int \mathcal{D}\phi\ \exp\bigl(-S(\phi) + (\phi, J)\bigr)}{\int \mathcal{D}\phi\ \exp\bigl(-S(\phi)\bigr)}\,, \tag{1.3}$$

where $(\phi, J) = \int \mathrm{d}^D x\, \phi(x)\, J(x)$. The theory is understood to be in infinite volume.[3] If C_∞ is massive, and provided that $V(\phi)$ is well behaved,[4] then (1.3) can be analyzed by standard technology. The renormalization group comes into the play when C_∞ is massless. It reduces the computation of (1.3) to an infinite sequence of massive integrals.

In perturbation theory, the Schwinger functions are computed as sums of connected Feynman diagrams built with the propagator C_∞ and all the vertices $V_n(x_1, \ldots, x_n)$ contained in $V(\phi)$. Although we do not intend to perform a perturbation expansion, we will think of $V(\phi)$ as a perturbation of the free theory, defined by C_∞. The following notation, which is widely spread in the mathematical literature, is adapted to this point of view. The formal Lebesque measure $\mathcal{D}\phi$ can be combined with the quadratic action to a gaussian measure[5]

$$\mathrm{d}\mu_{C_\infty}(\phi) \;=\; \det\left(2\pi C_\infty\right)^{-\frac{1}{2}}\ \mathcal{D}\phi\ \exp\left\{-\frac{1}{2}\left(\phi, \frac{1}{C_\infty}\phi\right)\right\} \tag{1.5}$$

on field space $\{\phi : \mathbb{R}^D \to \mathbb{R}\}$. The propagator C_∞ is called the covariance of $\mathrm{d}\mu_{C_\infty}(\phi)$. Furthermore, we can trade $W(J)$ for the generating function $\mathcal{V}(\Psi)$ of the connected Schwinger functions, amputed by C_∞. It is given by

$$W(J) \;=\; \frac{1}{2}\,(J, C_\infty J) - \mathcal{V}\,(C_\infty J)\,. \tag{1.6}$$

All information in $W(J)$ is also contained in $\mathcal{V}(\Phi)$. This new generating function has the path integral representation

$$\exp\bigl(-\mathcal{V}(\Phi)\bigr) \;=\; \frac{\int \mathrm{d}\mu_{C_\infty}(\phi)\ \exp\bigl(-V(\Phi + \phi)\bigr)}{\int \mathrm{d}\mu_{C_\infty}(\phi)\ \exp\bigl(-V(\phi)\bigr)}\,. \tag{1.7}$$

Finally, we will abbreviate $Z(\phi) = \exp\bigl(-V(\phi)\bigr)$ and speak of $Z(\phi)$ as a perturbation[6] of the gaussian measure. To compute the Schwinger functions of the effective field

[3] Think of (1.3) as a lattice field theory on an infinite unit lattice with a renormalization group improved action.

[4] $V(\phi)$ should be approximately local. $V(\phi)$ should also be stable.

[5] A less formal definition of the gaussian measure is given by its Fourier transform

$$\int \mathrm{d}\mu_{C_\infty}(\phi)\ \exp\bigl(\mathrm{i}\,(\phi, J)\bigr) \;=\; \exp\left\{-\frac{1}{2}\,(J, C_\infty J)\right\} \tag{1.4}$$

rather than (1.4).

[6] $Z(\phi)$ will be normalized such that $Z(0) = 1$.

theory, the effective interaction has to be convoluted with a massless gaussian measure. The idea of the renormalization group is to decompose this problem into digestible pieces. The first step is to decompose C_∞ into two parts, a massive short distance part C_L and a massless long distance remainder. Thanks to the convolution formula for gaussian measures, (1.7) then becomes a double integral, one integral over short distance fluctuations and another integral over long distance fluctuations.

Consider the following particular decomposition, which uses a dilatation of the field. Let $C_\infty(x,y)$ be decaying like a certain power of $|x-y|$. Say, $C_\infty(x,y)$ is of the order $O\left(|x-y|^{2-D-\epsilon}\right)$ at $|x-y|\to\infty$. We assume a unit ultraviolet cutoff, built in as a smooth momentum space regulator, such that $C_\infty(x,y)$ is a regular function of $|x-y|$ at short distances. In this situation, the following renormalization group [3] is custom built to study the properties of (1.7). Let D_L be the dilatation operator (whose transposed is) given by $D_L^{\mathrm{T}}(f)(x) = L^{1-\frac{D+\epsilon}{2}}\, f\left(\frac{x}{L}\right)$. We decompose C_∞ into

$$C_L \;=\; C_\infty - D_L^{\mathrm{T}}\, C_\infty\, D_L\,. \tag{1.8}$$

The dilatation operator is made such that the power like asymptotic tails of C_∞ and its scaled version cancel each other. The result is a massive propagator $C_L(x,y)$ which is of the order $O\left(F\left(\frac{|x-y|}{L}\right)\right)$ at $|x-y|\to\infty$ with a rapidly decreasing function $F(t)$. (We will choose an exponential cutoff where $F(t)=\mathrm{e}^{-\frac{t^2}{4}}$.) For $C_L(x,y)$ to be short ranged, the asymptotic of $C_\infty(x,y)$ and its scaled version $(D_L^{\mathrm{T}}\, C_\infty\, D_L)(x,y) = L^{2-D-\epsilon}\; C\left(\frac{x}{L},\frac{y}{L}\right)$ have to agree. This condition determines the scaling dimension in the dilatation operator D_L. One cannot decompose C_∞ with an anomalous dilatation, unless one gives up that C_L be short ranged.

The convolution formula for gaussian measures allows us to do the integral (1.7) in two steps. We have that

$$\exp\bigl(-\mathcal{V}(\Phi)\bigr) \;=\; \frac{\int \mathrm{d}\mu_{D_L C_\infty D_L^T}(\phi_{low})\;\int \mathrm{d}\mu_{C_L}(\phi_{high})\; Z(\Phi+\phi_{low}+\phi_{high})}{\int \mathrm{d}\mu_{D_L C_\infty D_L^T}(\phi_{low})\;\int \mathrm{d}\mu_{C_L}(\phi_{high})\; Z(\phi_{low}+\phi_{high})}\,. \tag{1.9}$$

The short distance covariance C_L has a unit ultraviolet cutoff and an infrared cutoff $1/L$ in momentum space. The long distance covariance has an ultraviolet cutoff $1/L$ and no infrared cutoff. Iterating this decomposition, one obtains a sequence of integrals over fluctuations on longer and longer scales. The special decomposition (1.8) is self similar in the sense that all fluctuation integrals obtained by iterating (1.9) are scaled versions of a single integral. Thus by rescaling ϕ_{low} with D_L, we obtain the transformation

$$R(C_L,D_L)(Z)(\Phi) = \frac{\int \mathrm{d}\mu_{C_L}(\phi)\; Z\bigl(D_L^{\mathrm{T}}(\Phi)+\phi\bigr)}{\int \mathrm{d}\mu_{C_L}(\phi)\; Z(\phi)} \tag{1.10}$$

of the perturbation $Z(\phi)$. It defines a representation of the semi-group of scale transformations with scale factor $L>1$ on a suitable space of perturbations $Z(\phi)$. (For $L<1$, C_L becomes negative.) Furthermore, it can be shown to perform a dilatation of the fully integrated potential $U(\Phi)$. Various applications of renormalization transformations of this kind are reviewed in [6].

The subject of this paper is to study general properties of renormalization group transformations of the form (1.10). The subject is not to compute the renormalization group flow for a particular model. To save space, I find it convenient to abbreviate (1.10) by (C_L, D_L). The meaning of this abbreviation is as follows. With the pair of operators (C_L, D_L), where C_L is a short ranged covariance, and where D_L is a dilatation, I associate the renormalization group transformation (1.10). The usefulness of this shorthand relies on the fact that the composition of two transformations of the form (1.10) is another transformation of this form.

We will show that the transformation given by (C_L, D_L) is equivalent to the transformation given by another pair (c_L, d_L). The equivalence is of the form

$$\begin{tikzcd} Z(\phi) \arrow[d, "R(C_L,D_L)"'] & z(\phi) \arrow[l, "Q(K_\star)"'] \arrow[d, "R(c_L,d_L)"] \\ Z(\phi|L) & z(\phi|L) \arrow[l, "Q(K_\star)"] \end{tikzcd} \tag{1.11}$$

where $Q(K_\star)(z)(\phi) = Z_\star(\phi)\ z(\phi)$ and $Z_\star(\phi) = \mathrm{e}^{-\frac{1}{2}(\phi, K_\star^{-1}\phi)}$ is a short ranged gaussian fixed point of $R_L(C_L, D_L)$. The new pair (c_L, d_L) defines a scale decomposition $c_\infty = d_L^{\mathrm{T}}\, c_\infty\, d_L + c_L$ of a covariance c_∞. The two formulations will be called the massless picture and the massive picture respectively. They are well known in the hierarchical (or ultra-local) approximation [7]. In the massive picture, both c_L and c_∞ are short ranged. Therefore, the scaling dimension of d_L is variable. In the massive picture, we will show the transformation defined by yet another pair (γ_L, δ_L), which defines a global (space-time independent) scale decomposition of c_∞, with δ_L the multiplication by a global factor L^{-1}, is a symmetry. Renormalization group and symmetry together spread a perturbation $z(\phi)$ to a surface

$$\begin{tikzcd} z(\phi) \arrow[r, "R(\gamma_{L'},\delta_{L'})"] \arrow[d, "R(c_L,d_L)"'] & z(\phi|1,L') \arrow[d, "R(c_L,d_L)"] \\ z(\phi|L,1) \arrow[r, "R(\gamma_{L'},\delta_{L'})"'] & z(\phi|L,L') \end{tikzcd} \tag{1.12}$$

We will then use the equivalence to translate the symmetry back to the massless picture. The composition of a renormalization group transformation, which scales by a factor L, and a symmetry transformation, which scales by a factor $L^{\frac{\eta}{2}}$, defines an anomalous scale decomposition of c_∞.

Inspired by the construction of a non-trivial $\phi^4_{4,\epsilon}$ fixed point in the planar approximation [8] and beyond it [5], we consider the renormalization group given by $C_\infty = F(-\triangle, \epsilon)\ (-\triangle)^{-1+\frac{\epsilon}{2}}$, where ϵ is a small parameter, and where $F(-\triangle, \epsilon)$ is a certain cutoff function. We will argue that the two cases $\epsilon = 0$ and $\epsilon \neq 0$ are different.

We will briefly discuss the relation between the anomalous dimension η and the symmetry applying the concepts of [9] to this case. The anomalous dimension of the $\phi^4_{4,\epsilon}$ fixed point is by definition zero.

2. Renormalization group – massless picture

We consider renormalization groups of transformations $R(C_L, D_L)$, defined by a pair of operators (C_L, D_L), where C_L is a covariance,[7] D_L is a dilatation, and L is a scale, with $L > 1$. Our starting point is a particular pair (C_L, D_L), which defines the renormalization group in the massless picture.

2.1. Covariance C_L

We start from a scale decomposition of the form (1.8). Let C_L be the covariance operator, acting as a convolution $C_L(f)(x) = \int \mathrm{d}^D y\, C_L(x,y)\, f(y)$ in real space, whose kernel given by the fourier integral

$$C_L(x,y) \;=\; \int \frac{\mathrm{d}^D p}{(2\pi)^D}\, \mathrm{e}^{\mathrm{i}\,p\,(x-y)}\, \widehat{C_L}(p)\,, \tag{2.1}$$

where $\widehat{C_L}(p)$ is a scaled difference of propagators

$$\widehat{C_L}(p) \;=\; \frac{\widehat{\chi}(p) - \widehat{\chi}(L\,p)}{(p^2)^{1-\frac{\epsilon}{2}}}\,. \tag{2.2}$$

Here $\widehat{\chi}(p)$ is a cutoff function, and ϵ is a small parameter (possibly zero). At $\epsilon = 0$, the free theory corresponding to (2.1) is simply a regularized massless free field. At $\epsilon \neq 0$, it is a statistical field theory with long ranged kinetic term. We choose a euclidean invariant regulator $\widehat{\chi}(p) = F(p^2)$. The function $F(s)$ could be a general monotonously decreasing function, which takes values between $F(0) = 1$ and $\lim_{s\to\infty} F(s) = 0$. For definiteness, we choose the smooth regulator, given by the incomplete Γ-function

$$\widehat{\chi}(p) \;=\; \frac{\Gamma\left(1-\frac{\epsilon}{2}, p^2\right)}{\Gamma\left(1-\frac{\epsilon}{2}\right)}. \tag{2.3}$$

For $\epsilon = 0$, it becomes an exponential cutoff: $\widehat{\chi}(p) = \mathrm{e}^{-p^2}$. The advantages of this particular cutoff function are apparent from the integral representation

$$\widehat{C_L}(p) \;=\; \frac{\int_1^{L^2} \frac{\mathrm{d}\alpha}{\alpha}\, \alpha^{1-\frac{\epsilon}{2}}\, \mathrm{e}^{-\alpha\, p^2}}{\Gamma\left(1-\frac{\epsilon}{2}\right)}. \tag{2.4}$$

Therefrom, the real space kernel is immediately computed as a gaussian integral and shown to decay exponentially. It reads

$$C_L(x,y) \;=\; \frac{\int_{L^{-2}}^1 \frac{\mathrm{d}\alpha}{\alpha}\, \alpha^{\frac{D+\epsilon}{2}-1}\, \mathrm{e}^{-\alpha\left(\frac{x-y}{2}\right)^2}}{(4\pi)^{\frac{D}{2}}\, \Gamma\left(1-\frac{\epsilon}{2}\right)}. \tag{2.5}$$

[7]The name covariance will be used as a synonym for a free propagator.

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The covariance C_L is thus a two sided regularization of $(-\triangle)^{-1+\frac{\epsilon}{2}}$, with unit ultraviolet cutoff and infrared cutoff L^{-1} (in units of mass). We included a parameter ϵ for the sake of generality, and to distinguish it from another parameter, the anomalous dimension η to appear below. We will assume that $D+\epsilon > 2$, so that (2.5) remains integrable in the infrared limit $L \to \infty$.

2.2. Dilatation D_L

When the infrared cutoff is removed, (2.5) becomes a massless covariance $C_\infty = \lim_{L\to\infty} C_L$ with a unit ultraviolet cutoff. In this limit, (2.4) simplifies to

$$\widehat{C_\infty}(p) \;=\; \frac{\Gamma\left(1-\frac{\epsilon}{2}, p^2\right)}{\Gamma\left(1-\frac{\epsilon}{2}\right)\;(p^2)^{1-\frac{\epsilon}{2}}}. \tag{2.6}$$

It follows that $C_\infty(x,y)$ decays powerlike at $|x-y| \to \infty$. Indeed, (2.5) becomes

$$C_\infty(x,y) \;=\; \frac{\gamma\left(\frac{D+\epsilon}{2}-1, \left(\frac{x-y}{2}\right)^2\right)}{(4\pi)^{\frac{D}{2}}\;\Gamma\left(1-\frac{\epsilon}{2}\right)\;\left|\frac{x-y}{2}\right|^{D+\epsilon-2}}. \tag{2.7}$$

Conversely, C_L is the difference of C_∞ and a scaled version of C_∞. Equation (2.2) reads $\widehat{C_L}(p) \;=\; \widehat{C_\infty}(p) - L^{2-\epsilon}\,\widehat{C_\infty}(L\,p)$. In real space, we therefore find that

$$C_L(x,y) \;=\; C_\infty(x,y) - L^{2-D-\epsilon}\,C_\infty\left(\frac{x}{L},\frac{y}{L}\right). \tag{2.8}$$

To save space, we will use the operator notation $C_\infty = F(-\triangle)/(-\triangle)^{1-\frac{\epsilon}{2}}$. Then (2.8) becomes $C_L = C_\infty - D_L^{\mathrm{T}}\,C_\infty\,D_L$, where D_L denotes the dilatation operator

$$D_L(f)(x) \;=\; L^{1+\frac{D-\epsilon}{2}}\,f(L\,x)\,, \tag{2.9}$$

and where D_L^{T} denotes its transposed

$$D_L^{\mathrm{T}}(f)(x) \;=\; L^{1-\frac{D+\epsilon}{2}}\,f\left(\frac{x}{L}\right). \tag{2.10}$$

In other words, D_L represents a dilatation by the scale factor L on function space $\{f : \mathbb{R}^D \to \mathbb{R}\}$, with scaling dimension $\sigma = 1 + \frac{D-\epsilon}{2}$.[8] To summarize, we have a pair of operators (C_L, D_L), depending on a scale factor L, where

1. D_L is a dilatation, with $D_1 = 1$, $D_L\,D_{L'} = D_{L\,L'}$, and

2. C_L is a short ranged covariance, with $C_L = C_\infty - D_L^{\mathrm{T}}\,C_\infty\,D_L$.

In this situation, we say that (C_L, D_L) defines a scale decomposition of C_∞ with respect to D_L.

[8]The dilatations by scale factors $L > 1$ form a semi-group.

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2.3. Transformation $R(C_L, D_L)$

As explained in the introduction, we associate a renormalization group transformation $R(C_L, D_L)$ with the pair (C_L, D_L). It consists of a gaussian convolution with respect to C_L, composed with a dilatation with respect to D_L:

$$R(C_L, D_L)(Z)(\Phi) = \frac{\int \mathrm{d}\mu_{C_L}(\phi)\ Z(D_L^{\mathrm{T}}(\Phi) + \phi)}{\int \mathrm{d}\mu_{C_L}(\phi)\ Z(\phi)} . \tag{2.11}$$

Here $\mathrm{d}\mu_{C_L}(\phi)$ denotes the gaussian measure with mean zero and covariance C_L, defined by $\int \mathrm{d}\mu_{C_L}(\phi)\ \mathrm{e}^{(\phi,f)} = \mathrm{e}^{\frac{1}{2}(f,C_L f)}$. The transformation acts on perturbations $Z(\phi)$, which will be normalized such that $Z(0) = 1$.

2.3.1. Physical meaning of $R(C_L, D_L)$

Admit a brief detour around the physical meaning of (2.11). We consider effective quantum field theories defined by the covariance C_∞ and an interaction $V(\phi) = -\log Z(\phi)$. Their physical content is coded in the generating function

$$\frac{\int \mathrm{d}\mu_{C_\infty}(\phi)\ Z(\phi)\ \mathrm{e}^{(\phi,J)}}{\int \mathrm{d}\mu_{C_\infty}(\phi)\ Z(\phi)} \ = \ \mathrm{e}^{\frac{1}{2}(J,C_\infty J)}\ \mathcal{Z}\big(C_\infty(J)\big) , \tag{2.12}$$

where $\mathcal{Z}(\Phi)$ [9] is given by a convolution of $Z(\phi)$ with $\mathrm{d}\mu_{C_\infty}(\phi)$, divided by a normalization constant:

$$\mathcal{Z}(\Phi) = \frac{\int \mathrm{d}\mu_{C_\infty}(\phi)\ Z(\Phi + \phi)}{\int \mathrm{d}\mu_{C_\infty}(\phi)\ Z(\phi)} . \tag{2.13}$$

The transformation $R(C_L, D_L)$ performs a dilatation of $\mathcal{Z}(\Phi)$: We have that

$$\mathcal{Z}(D_L^{\mathrm{T}}(\Phi)) = \frac{\int \mathrm{d}\mu_{C_\infty}(\phi)\ R(C_L, D_L)(Z)(\Phi + \phi)}{\int \mathrm{d}\mu_{C_\infty}(\phi)\ R(C_L, D_L)(Z)(\phi)} . \tag{2.14}$$

The transformation $R(C_L, D_L)$ thus enables a study of the behavior of $\mathcal{Z}(\Phi)$ under D_L^{T}, without having to perform the convolution with respect to $\mathrm{d}\mu_{C_\infty}(\phi)$, which is difficult because C_∞ is long ranged. If $R(C_L, D_L)(Z_\star)(\Phi) = Z_\star(\Phi)$ then $\mathcal{Z}\big(D_L^{\mathrm{T}}(\Phi)\big) = \mathcal{Z}(\Phi)$. In other words, we rediscover the well known fact that the correlators become dilatation invariant at a fixed point of the renormalization group.

2.3.2. Semi-group property of $R(C_L, D_L)$

We are lead to consider the following structure. We have a pair of operators (A, B) (with certain properties). With this pair, we associate a functional transformation

$$R(A, B)(Z)(\Phi) = \frac{\int \mathrm{d}\mu_A(\phi)\ Z(B^{\mathrm{T}}(\Phi) + \phi)}{\int \mathrm{d}\mu_A(\phi)\ Z(\phi)} . \tag{2.15}$$

[9] $\mathcal{Z}(\Phi)$ equals $\exp\big(-\mathcal{V}(\Phi)\big)$ and is the generating function of the free propagator amputated Schwinger functions.

The particular pair $(A,B)=(0,1)$ gives the identity transformation. Since $(C_1,D_1)=(0,1)$, we have that $R(C_1,D_1)=1$. The composition of two transformation (2.15) is another transformation of this kind. Let (A,B) and (A',B') be two pairs of operators. A standard computation with gaussian integrals reveals that

$$R(A,B)\;R(A',B')=R(B'^{\mathrm{T}}\,A\,B+A',B\,B')\,. \tag{2.16}$$

Equation (2.16) can be interpreted as a product $(A,B)(A',B')=(B'^{\mathrm{T}}\,A\,B+A',B\,B')$ of (A,B) and (A',B'). From the two properties $D_L\,D_{L'}=D_{L\,L'}$ and $C_L=C_\infty-D_L^{\mathrm{T}}\,C_\infty\,D_L$, it follows that $(C_L,D_L)\;(C_{L'},D_{L'})=(C_{L\,L'},D_{L\,L'})$. Therefore, the transformation $R(C_L,D_L)$ satisfies the semi-group property

$$R(C_L,D_L)\;R(C_{L'},D_{L'})=R(C_{L\,L'},D_{L\,L'})\,. \tag{2.17}$$

Of course, this does not come as a surprise. I included this computation to emphasize that the semi-group property can be traced back to the algebra of the pair of operators defining the renormalization group. Below, we will meet different pairs of operators which all satisfy this same algebra. Therefore all of them define semi-groups of renormalization transformations.

2.4. Generator $\dot{R}(\dot{C},\dot{D})$

The above formulas have a differential pendant. The discrete transformation $R(C_L,D_L)$ associates with a given initial perturbation $Z(\phi)$ a renormalization group trajectory $Z(\phi|L)=R(C_L,D_L)(Z)(\phi)$. Due to the semi-group property, $Z(\phi|L)$ is the solution of the discrete flow equation $R(C_L,D_L)(Z)(\phi|L')=Z(\phi|L\,L')$ to the initial condition $Z(\phi|1)=Z(\phi)$. The discrete flow equation implies a continuous flow equation for $Z(\phi|L)$, the renormalization group differential equation

$$L\frac{\partial}{\partial L}Z(\phi|L)=\dot{R}(\dot{C},\dot{D})(Z)(\phi|L)\,, \tag{2.18}$$

where $\dot{R}(\dot{C},\dot{D})=\frac{\partial}{\partial L}\big|_{L=1}R(C_L,D_L)$ is the generator of the renormalization group. Its explicit form is

$$\begin{aligned}L\frac{\partial}{\partial L}Z(\phi|L)=&\left[\frac{1}{2}\left(\frac{\delta}{\delta\phi},\dot{C}\frac{\delta}{\delta\phi}\right)+\left(\dot{D}^{\mathrm{T}}\phi,\frac{\delta}{\delta\phi}\right)\right]Z(\phi|L)-\\&-Z(\phi|L)\left[\frac{1}{2}\left(\frac{\delta}{\delta\phi},\dot{C}\frac{\delta}{\delta\phi}\right)+\left(\dot{D}^{\mathrm{T}}\phi,\frac{\delta}{\delta\phi}\right)\right]_{\phi=0}Z(\phi|L)\,,\end{aligned} \tag{2.19}$$

where $\dot{C}=\frac{\partial}{\partial L}\big|_{L=1}C_L$ and $\dot{D}^T=\frac{\partial}{\partial L}\big|_{L=1}D_L^{\mathrm{T}}$. See [10, 1]. The second term in (2.19) serves to maintain the normalization condition $Z(0)=1$.

In practical computations, $Z(\phi)$ is inconvenient due to its bad locality properties. There one prefers to work with the interaction $V(\phi|L)=-\log Z(\phi|L)$. Equation (2.19) immediately implies that

$$
\begin{aligned}
L\frac{\partial}{\partial L}V(\phi|L) = & \left[\frac{1}{2}\left(\frac{\delta}{\delta\phi},\dot{C}\frac{\delta}{\delta\phi}\right) + \left(\dot{D}^{\mathrm{T}}\phi,\frac{\delta}{\delta\phi}\right)\right] V(\phi|L) - \\
& - \frac{1}{2}\left(\frac{\delta}{\delta\phi}V(\phi|L)\,,\dot{C}\,\frac{\delta}{\delta\phi}V(\phi|L)\right) - \\
& - E(\dot{C})(V(\cdot|L))\,,
\end{aligned} \tag{2.20}
$$

where $E(\dot{C})(V(\cdot|L))$ denotes the (field independent) normalization constant

$$
\begin{aligned}
E(\dot{C})(V(\cdot|L)) = \frac{1}{2}\Bigg[&\left(\frac{\delta}{\delta\phi},\dot{C}\,\frac{\delta}{\delta\phi}\right)V(\phi|L) - \\
& - \left(\frac{\delta}{\delta\phi}V(\phi|L),\dot{C}\,\frac{\delta}{\delta\phi}V(\phi|L)\right)\Bigg]_{\phi=0}.
\end{aligned} \tag{2.21}
$$

Indispensable intermediate volume cutoffs are here treated casually. From (2.20) one also deduces a flow equation for the 1PI-vertex functions. We will not repeat this calculation here. The bottom line is that with every scale decomposition (C_L, D_L) is associated an exact renormalization group differential equation. For the instant pair (C_L, D_L), the differential version $(\dot{C}, \dot{D})$ is explicitely given by

$$
\dot{C}(x,y) \;=\; \frac{2\,\mathrm{e}^{-\left(\frac{x-y}{2}\right)^2}}{(4\pi)^{\frac{D}{2}}\,\Gamma\left(1-\frac{\epsilon}{2}\right)}, \tag{2.22}
$$

together with

$$
\dot{D}^{\mathrm{T}}(f)(x) \;=\; \left[1-\frac{D+\epsilon}{2}-x\frac{\partial}{\partial x}\right]\,f(x)\,. \tag{2.23}
$$

Notice that the differential pair $(\dot{C}, \dot{D})$ is independent of L. This is a characteristic feature of the rescaled renormalization group with.

To summarize, the renormalization group associated with the pair of operators (C_L, D_L) comes in two equivalent formulations. In its discrete version, one deals with difference equations. In its continuous version, one deals with differential equations.

3. Renormalization group – massive picture

In this section, we show that the renormalization group defined by the pair (C_L, D_L) is equivalent to a renormalization group defined by another pair (c_L, d_L).

3.1. Gaussian fixed point

The transformation $R(C_L, D_L)$ has a gaussian fixed point[10] $Z_\star(\phi) = \mathrm{e}^{-\frac{1}{2}(\phi,K_\star^{-1}\phi)}$, given by $K_\star = (1-\chi)/(-\triangle)^{1-\frac{\epsilon}{2}}$. For the particular cutoff function (2.3), it has the integral

[10] $Z_\star(\phi)$ satisfies $R(C_L, D_L)(Z_\star)(\phi) \;=\; Z_\star(\phi)$ for all $L > 1$.

representation

$$K_\star(x,y) \;=\; \frac{\int_1^\infty \frac{\mathrm{d}\alpha}{\alpha}\,\alpha^{\frac{D+\epsilon}{2}-1}\,\mathrm{e}^{-\alpha\left(\frac{x-y}{2}\right)^2}}{(4\pi)^{\frac{D}{2}}\,\Gamma\left(1-\frac{\epsilon}{2}\right)}. \tag{3.1}$$

Inspecting (3.1), it follows that $K_\star$ is short ranged. The fourier transform of (3.1) is indeed regular at zero momentum: $\widehat{K_\star}(p) \;=\; 1/\Gamma\left(2-\frac{\epsilon}{2}\right)+O(p^2)$. More generally, we have a line of fixed points given by $K_\star + \mathrm{const.}/(-\triangle)^{1-\frac{\epsilon}{2}}$. All of them are long ranged, except for the special point with const. $= 0$.

3.2. Transformation $Q(C)$

The multiplication of perturbations with gauss functions and the renormalization group transformation satisfy the following commutation law. Let C be an operator. Let $Q(C)$ be the functional operator given by

$$Q(C)(Z)(\phi) = \mathrm{e}^{-\frac{1}{2}(\phi,C^{-1}\,\phi)}\;Z(\phi)\,. \tag{3.2}$$

Thus $Q(C)$ multiplies a perturbation with a Gauss function. The two transformations $R(A,B)$ and $Q(C)$ conspire to the identity

$$R(A,B)\;Q(C) = Q(B^{-1\mathrm{T}}\,\mathcal{L}^{-1}\,C\,B^{-1})\;R(\mathcal{L}\,A,B\,\mathcal{L})\,, \tag{3.3}$$

where $\mathcal{L}^{-1} \;=\; 1 + A\,C^{-1}$, assuming that $A = A^{\mathrm{T}}$, $C = C^{\mathrm{T}}$, invertibility of A, B, and C, and existence of the gaussian convolutions. In other words, to commute a gauss function through the renormalization group transformation requires to compute its renormalization group image.

3.3. Substitution $Z(\phi) = Z_\star(\phi)\;z(\phi)$

Let us write the perturbation $Z(\phi)$ as a product of the fixed point $Z_\star(\phi)$ and a new perturbation $z(\phi)$. Substitute $Z(\phi) = Z_\star(\phi)\;z(\phi)$ to obtain

$$R(C_L,D_L)(Z_\star\;z)(\phi) = Z_\star(\phi)\;R(c_L,d_L)(z)(\phi)\,, \tag{3.4}$$

with a new pair (c_L,d_L) of operators, defining the massive picture of the renormalization group. They are given by

$$c_L = \mathcal{L}_L\,C_L, \quad d_L = D_L\,\mathcal{L}_L\,, \tag{3.5}$$

where $\mathcal{L}_L$ is the operator $\mathcal{L}_L = K_\star/(K_\star+C_L)$. Notice that C_L and $K_\star$ commute. $\mathcal{L}_L$ is explicitely given by

$$\widehat{\mathcal{L}_L}(p) \;=\; \frac{1-\widehat{\chi}(p)}{1-\widehat{\chi}(L\,p)} \;=\; L^{-2+\epsilon}\,\frac{\widehat{K_\star}(p)}{\widehat{K_\star}(L\,p)}\,. \tag{3.6}$$

Think of $\widehat{\mathcal{L}_L}(p)$ as a p-dependent correction to the scale factor in the field dilatation. At $p^2 \;=\; 0$ it yields an extra factor $L^{-2+\epsilon}$ to the dilataton, while it becomes one as $p^2 \to \infty$. Remarkably, we obtain another renormalization group transformation of the same kind as before. But the covariance and the dilatation have changed.

3.4. Covariance c_L

Like C_L, the covariance c_L is short ranged. It is again diagonal in momentum space. From (3.6), it follows that

$$\widehat{c_L}(p) \;=\; \widehat{\mathcal{L}_L}(p)\,\widehat{C_L}(p) \;=\; \frac{1-\widehat{\chi}(p)}{1-\widehat{\chi}(L\,p)}\;\frac{\widehat{\chi}(p)-\widehat{\chi}(L\,p)}{(p^2)^{1-\frac{\epsilon}{2}}}\,. \tag{3.7}$$

The new covariance and the old covariance both vanish at $L = 1$. Therefore, the corresponding renormalization group transformations become the identity at $L = 1$ as they should. On the other side, (3.7) turns into

$$\widehat{c_\infty}(p) \;=\; \frac{\big(1-\widehat{\chi}(p)\big)\,\widehat{\chi}(p)}{(p^2)^{1-\frac{\epsilon}{2}}} \;=\; \big(1-\widehat{\chi}(p)\big)\,\widehat{C_\infty}(p)\,, \tag{3.8}$$

when $L \to \infty$. Recall that C_∞ is a massless covariance with unit ultraviolet cutoff. In contrast, c_∞ is a massive (short ranged) covariance with twosided unit cutoffs.

3.5. Dilatation d_L

From (3.6), it follows that $\mathcal{L}_1 = 1$ and consequently $d_1 = 1$, in analogy to $D_1 = 1$. Furthermore, d_L satisfies the composition law $d_L\; d_{L'} = d_{L\,L'}$. The pair (c_L, d_L) can be shown to be a scale decomposition of the covariance c_∞ with respect to d_L: We have that $c_L = c_\infty - d_L^{\mathrm{T}}\,c_\infty\,d_L$. Therefrom, we conclude that $R(c_L, d_L)$ satisfies the semi-group properties $R(c_1, d_1) = 1$ and

$$R(c_L, d_L)\;R(c_{L'}, d_{L'}) \;=\; R(c_{L\,L'}, d_{L\,L'})\,. \tag{3.9}$$

The reasoning is completely analogous to the massless case, since the algebraic properties of the pair (c_L, d_L) are identical to those of (C_L, D_L). We conclude that the transformation (3.4) is an equivalence between two representations of the renormalization group, a massless picture and a massive picture. The attributes massless and massive refer to the underlying scale decomposed covariance.

3.6. Generator $\dot{R}(\dot{c}, \dot{d})$

To compute the renormalization group differential equation in the massive picture, all we have to do, is to compute the differential pair $(\dot{c}, \dot{d})$ and to replace $(\dot{C}, \dot{D})$ by it in (2.22) and (2.23). The differential covariance and dilatation pick up an extra term $\dot{\mathcal{L}}$ as compared to the massless picture. The extra term is

$$\widehat{\dot{\mathcal{L}}}(p) \;=\; \frac{\partial}{\partial L}\bigg|_{L=1}\,\widehat{\mathcal{L}_L}(p) \;=\; \frac{p\nabla_p\widehat{\chi}(p)}{1-\widehat{\chi}(p)}. \tag{3.10}$$

With our cutoff function (2.3), the extra term becomes

$$\widehat{\dot{\mathcal{L}}}(p) \;=\; \frac{-2\,(p^2)^{1-\frac{\epsilon}{2}}\;\mathrm{e}^{-p^2}}{\gamma\left(1-\frac{\epsilon}{2}, p^2\right)} \;=\; \frac{-2}{\int_0^1 \mathrm{d}\alpha\;(1-\alpha)^{\frac{-\epsilon}{2}}\;\mathrm{e}^{\alpha\,p^2}}\,. \tag{3.11}$$

In the special case when $\epsilon = 0$, it becomes the generating function of the Bernoulli numbers, namely $\widehat{\dot{\mathcal{L}}}(p) = -2\,p^2/(\mathrm{e}^{p^2} - 1)$. In the massive picture, we thus have a differential pair of operators $(\dot{c}, \dot{d})$, given by

$$\widehat{\dot{c}}(p) \;=\; \widehat{\dot{\mathcal{L}}}(p) + \widehat{\dot{C}}(p) \;=\; 2\,\mathrm{e}^{-p^2}\left(\frac{1}{\Gamma\left(1-\frac{\epsilon}{2}\right)} - \frac{(p^2)^{1-\frac{\epsilon}{2}}}{\gamma\left(1-\frac{\epsilon}{2}, p^2\right)}\right), \tag{3.12}$$

together with

$$\widehat{\dot{d}^{\mathrm{T}}(f)}(p) \;=\; \widehat{\dot{\mathcal{L}}(f)}(p) + \widehat{\dot{D}^{\mathrm{T}}(f)}(p) \;=\; \left[\widehat{\dot{\mathcal{L}}}(p) + 1 + \frac{D-\epsilon}{2} + p\frac{\partial}{\partial p}\right]\,\widehat{f}(p)\,. \tag{3.13}$$

The change from the massless picture to the massive picture is thus particularly simple in the differential formulation of the renormalization group. Finally, we remark that $d_L = D_L\,\mathcal{L}_L\,D_L^{-1}\,D_L$ with

$$(\widehat{D_L\mathcal{L}_L D_L^{-1}})(p) \;=\; \widehat{\mathcal{L}_L}(L^{-1}p) \;=\; \frac{1-\widehat{\chi}(L^{-1}\,p)}{1-\widehat{\chi}(p)}\,. \tag{3.14}$$

Which of the two pictures is better suited, depends on the kind of perturbation one is considering, whether it is functionally closer to the trivial fixed point or to the gaussian fixed point. In the hierarchical approximation for instance, the massive picture has proved to be better suited for an investigation of the non-trivial scalar fixed point in three dimensions [7].

4. Symmetry – massive picture

In this section, we show that the renormalization group defined by the pair (c_L, d_L) has a symmetry, the transformation defined by yet another pair (γ_L, δ_L).

4.1. Symmetry pair (γ_L, δ_L)

Let L be a scale, with $L > 1$. Let γ_L and δ_L be the operators given by

$$\gamma_L = \left(1 - L^{-2}\right)\;c_\infty, \quad \delta_L = L^{-1}\,. \tag{4.1}$$

The operator δ_L performs a global (space-time independent) multiplication by L^{-1}. Obviously, (γ_L, δ_L) is a (global) scale decomposition of c_∞ with respect to δ_L, namely $\gamma_L = c_\infty - \delta_L^{\mathrm{T}}\,c_\infty\,\delta_L$. It follows that the transformation $R(\gamma_L, \delta_L)$ satisfies the semi-group properties $R(\gamma_1, \delta_1) = 1$ and

$$R(\gamma_L, \delta_L)\;R(\gamma_{L'}, \delta_{L'}) \;=\; R(\gamma_{L\,L'}, \delta_{L\,L'})\,. \tag{4.2}$$

4.2. Commutator of $R(c_L, d_L)$ and $R(\gamma_{L'}, \delta_{L'})$

The two one parameter families of transformations $R(c_L, d_L)$ and $R(\gamma_{L'}, \delta_{L'})$ commute: For all $L > 1$ and $L' > 1$, we have that

$$R(c_L, d_L)\; R(\gamma_{L'}, \delta_{L'}) = R(\gamma_{L'}, \delta_{L'})\; R(c_L, d_L)\,. \tag{4.3}$$

This property defines a symmetry of $R(c_L, d_L)$. An analogous construction can be made in the massless picture. The point here is that the symmetry covariance is short ranged in the massive picture. To prove (4.3), we compute

$$(c_L, d_L)\; (\gamma_{L'}, \delta_{L'}) = (\delta_{L'}^{\mathrm{T}}\, c_L\, \delta_{L'} + \gamma_{L'}, d_L\, \delta_{L'}) \tag{4.4}$$

and compare the result with

$$(\gamma_{L'}, \delta_{L'})\; (c_L, d_L) = (d_L^{\mathrm{T}}\, \gamma_{L'}\, d_L + c_L, \delta_{L'}\, d_L)\,. \tag{4.5}$$

The two dilatations commute, $d_L\, \delta_{L'} = \delta_{L'}\, d_L$. The combined covariances are

$$\begin{aligned}\delta_{L'}^{\mathrm{T}}\, c_L\, \delta_{L'} + \gamma_{L'} &= (L')^{-2} \left(c_\infty - d_L^{\mathrm{T}}\, c_\infty\, d_L\right) + \left(1 - (L')^{-2}\right) c_\infty \\ &= c_\infty - (L')^{-2}\, d_L^{\mathrm{T}}\, c_\infty\, d_L \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} d_L^{\mathrm{T}}\, \gamma_{L'}\, d_L + c_L &= \left(1 - (L')^{-2}\right) d_L^{\mathrm{T}}\, c_\infty\, d_L + c_\infty - d_L^{\mathrm{T}}\, c_\infty\, d_L \\ &= c_\infty - (L')^{-2}\, d_L^{\mathrm{T}}\, c_\infty\, d_L\,. \end{aligned} \tag{4.7}$$

These two covariances are identical. It follows that $R(c_L, d_L)$ and $R(\gamma_{L'}, \delta_{L'})$ indeed commute.

4.3. Composed transformation

To obtain a better understanding of the symmetry transformation, we can study the composition of a renormalization group transformation and a symmetry transformation. This composition turns out to be given by yet another pair. Let $L' = L^{\frac{\eta}{2}}$, where η parametrizes the symmetry transformation. (A special value of η will be identified with the anomalous dimension.) Then we have that

$$R(c_L, d_L)\, R(\gamma_{L^{\frac{\eta}{2}}}, \delta_{L^{\frac{\eta}{2}}}) = R(c_{L,\eta}, d_{L,\eta})\,, \tag{4.8}$$

where

$$d_{L,\eta} \;=\; d_L\, \delta_{L^{\frac{\eta}{2}}} \;=\; L^{-\frac{\eta}{2}}\, d_L \tag{4.9}$$

and

$$c_{L,\eta} \;=\; c_\infty - L^{-\eta}\, d_L^{\mathrm{T}}\, c_\infty\, d_L \;=\; c_\infty - d_{L,\eta}^{\mathrm{T}}\, c_\infty\, d_{L,\eta}\,. \tag{4.10}$$

In other words, the composition of the renormalization group and the symmetry is a renormalization group built from an anomalous scale decomposition $(c_{L,\eta}, d_{L,\eta})$ of c_∞.

4.4. Differential pair $(\dot{\gamma}, \dot{\delta})$

The generator of the symmetry is particularly simple. It is given by the differential pair of operators $(\dot{\gamma}, \dot{\delta})$ with $\dot{\gamma} = 2\, c_\infty$ and $\dot{\delta} = -1$. The generators of the renormalization group and the symmetry are

$$r \;=\; \frac{1}{2}\,\left(\frac{\delta}{\delta\phi}, \dot{c}\frac{\delta}{\delta\phi}\right) + \left(\dot{d}^{\mathrm{T}}\phi, \frac{\delta}{\delta\phi}\right) \tag{4.11}$$

and

$$u \;=\; \frac{1}{2}\,\left(\frac{\delta}{\delta\phi}, \dot{\gamma}\frac{\delta}{\delta\phi}\right) + \left(\dot{\delta}^{\mathrm{T}}\phi, \frac{\delta}{\delta\phi}\right). \tag{4.12}$$

They satisfy $[r,u]=0$, as can be checked directly. Renormalization group and symmetry associate with a perturbation $z(\phi)$ a two parametric orbit

$$z(\phi|L,L') \;=\; R(c_L, d_L)\, R(\gamma_{L'}, \delta_{L'})(z)(\phi)\,, \tag{4.13}$$

whose dependence on L is governed by (4.11), and whose dependence on L' is governed by (4.12). Indeed, (4.13) satisfies the two differential equations

$$L\frac{\partial}{\partial L} z(\phi|L,L') \;=\; r(z)(\phi|L,L') - z(\phi|L,L')\; r(z)(0|L,L') \tag{4.14}$$

and

$$L'\frac{\partial}{\partial L'} z(\phi|L,L') \;=\; u(z)(\phi|L,L') - z(\phi|L,L')\; u(z)(0|L,L')\,. \tag{4.15}$$

The interaction $v(\phi|L,L') = -\log z(\phi|L,L')$ obeys an exponentiated version of (4.14) and (4.15), analogous to (2.20). We leave it to the reader, to write it down explicitely.

5. Symmetry – massless picture

In this section, we transform the symmetry $R(\gamma_L, \delta_L)$ to the massless picture. Recall that the change of pictures is the transformation $Z(\phi) = Q(K_\star)(z)(\phi)$ and that

$$R(C_L, D_L)\; Q(K_\star) = Q(K_\star)\; R(c_L, d_L)\,. \tag{5.1}$$

To translate $R(\gamma_L, \delta_L)$ to the massless picture, we have to conjugate with $Q(K_\star)$. From (4.8) and

$$Q(C)\; Q(D) = Q\left(C\frac{1}{C+D}D\right) \tag{5.2}$$

it follows that

$$R(A,B)\; Q(C) = Q(C)\; Q\left(-C\frac{1}{C'-C}C'\right)\; R(A',B')\,, \tag{5.3}$$

with $A' = \mathcal{L}\; A$, $B' = B\;\mathcal{L}$, $C' = (B^{-1})^{\mathrm{T}}\;(C+A)\;B^{-1}$, where $\mathcal{L} = 1/(1 + A\,C^{-1})$. In the massless picture, the symmetry transformation therefore consists of two parts. One part is the transformation given by a pair of operators. The second part is the multiplication with a certain Gauss function. This term is essentially a kinetic term as we will see.

5.1. A', B', C', and C'' for the symmetry

Let A, B, and C be given by $A = (1 - L^{-2})\ c_\infty$, $B = L^{-1}$, and $C = -K_\star$. It follows that $\mathcal{L} = 1/(1 - (1 - L^{-2})\,\chi)$ and thus

$$A' = \frac{(1 - L^{-2})\ c_\infty}{1 - (1 - L^{-2})\chi}\,, \tag{5.4}$$

$$B' = \frac{L^{-1}}{1 - (1 - L^{-2})\chi}, \tag{5.5}$$

$$C' = \left(L^2 - 1 - \frac{L^2}{\chi}\right)\ c_\infty\,, \tag{5.6}$$

and

$$\frac{1}{C''} = \frac{C - C'}{C\,C'} = \frac{(1 - L^{-2})\ (-\triangle)^{1-\frac{\epsilon}{2}}}{1 - (1 - L^{-2})\chi}\,. \tag{5.7}$$

The symmetry transformation $R(\gamma_L, \delta_L)$ thus has the following form in the massless picture. It maps $Z(\phi)$ to a symmetry orbit

$$Z(\Phi|1, L) = \mathrm{e}^{-\frac{1}{2}(\Phi, \frac{1}{C''}\Phi)}\ \frac{\int \mathrm{d}\mu_{A'}(\phi)\ Z((B')^{\mathrm{T}}(\Phi) + \phi)}{\int \mathrm{d}\mu_{A'}(\phi)\ Z(\phi)}\,. \tag{5.8}$$

Remarkably, the function $\widehat{C''}(p)^{-1} = (1 - L^{-2})\ (p^2)^{1-\frac{\epsilon}{2}}/\big(1 - (1 - L^{-2})\,\widehat{\chi}(p)\big)$ is an analytic function of p^2 at the origin only if $\epsilon = 0$. A straight forward calculation confirms directly that the transformation (5.8) satisfies the properties of a semi-group and that it commutes with the renormalization group.

5.2. A', B', C', and C'' for the composed transformation

The composition of a symmetry transformation, which scales by a factor L', and a renormalization group transformation, which scales by a factor L, is computed analogously. Let $A = c_\infty - (L')^{-2}\, d_L^{\mathrm{T}}\, c_\infty\, d_L$, $B = (L')^{-1}\, d_L$, and $C = -K_\star$. Then

$$\mathcal{L} = \frac{1 - \chi_L}{1 - \chi}\ \frac{1}{1 - \big(1 - (L')^{-2}\chi_L\big)}\,, \tag{5.9}$$

where $\widehat{\chi_L}(p) = \widehat{\chi}(L\,p)$. It follows that

$$A' = \frac{1}{(-\triangle)^{1-\frac{\epsilon}{2}}}\ \frac{\chi\,(1 - \chi_L) - (L')^{-2}(1 - \chi)\,\chi_L}{1 - \big(1 - (L')^{-2}\big)\,\chi_L}\,, \tag{5.10}$$

$$B' = D_L\ \frac{(L')^{-1}}{1 - \big(1 - (L')^{-2}\big)\chi_L}\,, \tag{5.11}$$

$$C' = \frac{1}{(-\triangle)^{1-\frac{\epsilon}{2}}}\ (1 - \chi)\ (L')^2\ \Big(-1 + \big(1 - (L')^{-2}\big)\,\chi\Big)\,, \tag{5.12}$$

and

$$\frac{1}{C''} = \frac{1-(L')^{-2}}{1-\big(1-(L')^{-2}\big)\chi}\,(-\triangle)^{1-\frac{\epsilon}{2}}\,. \tag{5.13}$$

For $L = 1$, one reproduces the formulas (5.4), (5.5), and (5.6) for the pure symmetry transformation. Notice that (5.13) is independent of L, as it should. Equation (5.8) generalizes to

$$Z(\Phi|L,L') = \mathrm{e}^{-\frac{1}{2}(\Phi,\frac{1}{C''}\Phi)}\,\frac{\int \mathrm{d}\mu_{A'}(\phi)\; Z((B')^{\mathrm{T}}(\Phi)+\phi)}{\int \mathrm{d}\mu_{A'}(\phi)\; Z(\phi)}\,. \tag{5.14}$$

5.3. Generators in the massless picture

We can compute the generators of the double flow (5.14) for the interaction $V(\phi|L,L') = -\log Z(\phi|L,L')$ from

$$\begin{aligned} V(\Phi|L,L') = {} & \frac{1}{2}\left(\Phi,\frac{1}{C''}\Phi\right) - \\ & -\log\int \mathrm{d}\mu_{A'}(\phi)\;\mathrm{e}^{-V\left((B')^{\mathrm{T}}(\Phi)+\phi\right)} + \\ & +\log\int \mathrm{d}\mu_{A'}(\phi)\;\mathrm{e}^{-V(\phi)}\,, \end{aligned} \tag{5.15}$$

where A', B', and $\frac{1}{C''}$ are given by (5.10), (5.11), and (5.13) respectively. We find two flow equations. The first flow equation is identical with (2.20). It describes the flow in the renormalization group direction. The second flow equation is

$$\begin{aligned} L'\frac{\partial}{\partial L'}V(\phi|L,L') = {} & \big(\phi,(-\triangle)^{1-\frac{\epsilon}{2}}\phi\big) + \\ & +\left[\frac{1}{2}\left(\frac{\delta}{\delta\phi},2\,c_\infty\,\frac{\delta}{\delta\phi}\right) + \left((-1+2\chi)\phi,\frac{\delta}{\delta\phi}\right)\right] V(\phi|L,L') - \\ & -\frac{1}{2}\left(\frac{\delta}{\delta\phi}V(\phi|L,L'),2\,c_\infty\frac{\delta}{\delta\phi}V(\phi|L,L')\right) - \\ & -E(V(\cdot|L,L')), \end{aligned} \tag{5.16}$$

where

$$\begin{aligned} E(V(\cdot|L,L')) = {} & \frac{1}{2}\Bigg[\left(\frac{\delta}{\delta\phi},2\,c_\infty\frac{\delta}{\delta\phi}V(\phi|L,L')\right) - \\ & -\left(\frac{\delta}{\delta\phi}V(\phi|L,L'),2\,c_\infty\frac{\delta}{\delta\phi}V(\phi|L,L')\right)\Bigg]_{\phi=0}\,. \end{aligned} \tag{5.17}$$

It describes the flow in the symmetry direction. Both together, plus an initial condition, are equivalent to the integral (5.15). The novelty as compared to the massive picture is the appearance of a kinetic term.

6. Anomalous dimension η

This section contains a brief discussion of fixed points with anomalous dimension based on the following hypothesis, adapted from [9] and [4]. See also the recent discussion in [11, 12] and references therein.

6.1. Fixed point $(z_\star(\phi), \eta)$

We define a fixed point $z(\phi)$ (in the massive picture) with anomalous dimension η as a fixed point of $R(c_L, d_L)\, R(\gamma_{L^{\frac{\eta}{2}}}, \delta_{L^{\frac{\eta}{2}}})$, the composition of renormalization group transformation $R(c_L, d_L)$, which scales by a factor L, and a symmetry transformation $R(\gamma_{L^{\frac{\eta}{2}}}, \delta_{L^{\frac{\eta}{2}}})$, which scales by a factor $L^{\frac{\eta}{2}}$.

6.2. Discussion of $(z_\star(\phi), \eta)$

$z_\star(\phi)$ is a fixed point of the renormalization group transformation $R(c_L, d_L)$ only if $\eta = 0$. In this case, there are two possibilities. Either $z_\star(\phi)$ is also a fixed point of $R(\gamma_{L'}, \delta_{L'})$, so to speak a double fixed point, or one has a line of fixed points $z_\star(\phi|L')$ generated as a symmetry orbit from an arbitrary representative $z_\star(\phi)$. This is the case for the trivial fixed point.

The case with anomalous dimension η is a generalization hereof. There one has an invariant line of the renormalization group rather than a line of fixed points, where the renormalization group acts on the invariant line by an inverse symmetry transformation. A fixed point $z_\star(\phi)$ with non-zero anomalous dimension η is *not* a stationary renormalization group flow. (There remains the possibility to declare a composition of the renormalization group and the symmetry to be a new renormalization group.) The appropriate framework to analyze such a situation is to divide the space of perturbations into symmetry orbits and to consider the renormalization group on orbit space. A natural possibility is to consider one representative on each orbit by imposing a renormalization condition, which breaks the symmetry. Such a condition could be to demand the pre factor of the kinetic term in the interaction to be a given number. In the presence of a symmetry, the spectrum of a fixed point should include a marginal operator, the direction of the symmetry.

A principal problem is the following. To make the fixed point problem tractable, one has to truncate the space of interactions. Ideally, the truncation would be such that the symmetry leaves invariant the space of truncated interactions. Because then, the orbit construction can be done on the truncated space. If however the truncation breaks the symmetry, the topic of anomalous dimension is buried in the no-truncation limit, which is usually very difficult to analyze. The idea forwarded in [9] for this situation is to truncate in such a way that a marginal operator survives truncation. In other words that the truncations is such that at least differentially the symmetry persists. The local approximation to the renormalization group unfortunately breaks the symmetry, wherefore the concept of anomalous dimension loses its meaning in the local approximation, and in particular in the hierarchical renormalization group.

6.3. ϵ-model

Consider the fixed point constructions [8] and [5] for the ϵ-model in this light. Their constructions work directly in the massless picture. When ϵ is non-zero, the situation is the following. If one restricts the renormalization group to vertices, which are $\mathcal{C}^\infty$-functions of the momenta, then the symmetry can be ruled out because it generates a non-$\mathcal{C}^\infty$ kinetic term. In this situation, the authors consider an invariant subspace of the renormalization group, which is not invariant under the symmetry. Remarkably, the authors succeed to construct a non-trivial fixed point in the $\mathcal{C}^\infty$-subspace. (For the planar approximation, this is perhaps not so surprising because the planar approximation itself also breaks the symmetry.) Such a non-trivial fixed point is by definition a fixed point with anomalous dimension zero.

If on the other hand one wants to investigate flows modulo the symmetry (as one has to in the case of non-zero anomalous dimension) then one cannot restrict the flow to $\mathcal{C}^\infty$-vertices in this model. Whether the ϵ-model has another fixed point in this bigger space, I do not know. If this is the case, it would be very interesting to determine $\eta(\epsilon)$. In the most interesting case, the three dimensional theory with $\epsilon = 0$, the situation is different. There the symmetry preserves the property of analyticity in the momenta. Therefore it cannot be ruled out.

6.4. Differential equation for $v_\star(\phi)$

The fixed point problem (6.1) in the massive picture takes the form of the following renormalization group differential equation for the fixed point interaction $v_\star(\phi) = -\log z_\star(\phi)$,

$$\begin{aligned}&\left[\frac{1}{2}\left(\frac{\delta}{\delta\phi}, [\dot{c} - \eta\, c_\infty]\frac{\delta}{\delta\phi}\right) + \left(\left[\dot{d}^{\mathrm{T}} - \frac{\eta}{2}\right]\phi, \frac{\delta}{\delta\phi}\right)\right] v_\star(\phi) \\ &-\frac{1}{2}\left(\frac{\delta}{\delta\phi}v_\star(\phi), [\dot{c} - \eta\, c_\infty]\,\frac{\delta}{\delta\phi}v_\star(\phi)\right) = E(\dot{c} - \eta\, c_\infty)(v_\star)\,,\end{aligned} \tag{6.1}$$

where

$$\begin{aligned}E(\dot{c} - \eta\, c_\infty)(v_\star) = \frac{1}{2}\Bigg[&\left(\frac{\delta}{\delta\phi}, [\dot{c} - \eta\, c_\infty]\,\frac{\delta}{\delta\phi}\right) v(\phi) \\ &- \left(\frac{\delta}{\delta\phi}v_\star(\phi), [\dot{c} - \eta\, c_\infty]\,\frac{\delta}{\delta\phi}v_\star(\phi)\right)\Bigg]_{\phi=0}.\end{aligned} \tag{6.2}$$

A natural problem is to find good truncation schemes for (6.1) and (6.2) and to determine η by the condition that the spectrum of $v_\star(\phi)$ have a marginal operator. An investigation of various schemes will be presented elsewhere.

Recall the generators (4.11) and (4.12) of the renormalization group and the symmetry respectively. If we neglect the trivial normalization constants, the fixed point problem becomes a linear eigenvalue problem

$$r(z_\star)(\phi) \;=\; -\frac{\eta}{2}\, u(z_\star)(\phi) \tag{6.3}$$

for $r\, u^{-1}$. The meaning of the anomalous dimension is thus to be an eigenvalue. Equation (6.1) is just an exponentiated and thus non-linear form of (6.3). The analogous equations in the massless picture follow immediately from (5.15) and (5.16).

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